

The LIBINT Programmer's Manual

Edward F. Valeev

Department of Chemistry, Virginia Tech, Blacksburg, Virginia 24061 USA

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1 Introduction

LIBINT library contains functions to compute many-body integrals over Gaussian functions which appear in electronic and molecular structure theories. LIBINT Version 2.9.0 ()¹ can currently compute several different types of integrals:

- Two-body Coulomb (electron repulsion) integrals (ERIs). This is by far the most common type of integrals in molecular structure theory. Two-, three-, and four-center integrals, and their geometrical derivatives, are supported.
- Two-body integrals which appear in explicitly correlated R12/F12 methods with Gaussian correlation factors.²⁻⁴ All R12 methods, such as MP2-R12, contain terms in the wave function that depend on the interelectronic distances r_{ij} (hence the name). Appearance of several types of two-body integrals is due to the use of the approximate resolution of the identity to reduce three- and four-body integrals to products of simpler integrals.

A somewhat unusual feature of LIBINT library is that its source code is generated by a computer program, i.e., a *compiler*. The purpose of the LIBINT compiler is twofold:

- First, the compiler eliminates need for tedious writing, debugging, and optimizing the integrals code. Instead, a programmer provides high-level specification of operators and recurrence relations, and heuristics of how to apply the recurrence relations and the compiler takes care of the rest. This currently requires some high-level programming, in a domain-specific mini-language tailored for expressing recurrence relations. For example, the following Obara-Saika recurrence relation,⁵

$$\begin{aligned}
 (\mathbf{a} + 1_i \mathbf{b} | \mathbf{c} \mathbf{d})^{(m)} &= (\mathbf{P}\mathbf{A})_i (\mathbf{a} \mathbf{b} | \mathbf{c} \mathbf{d})^{(m)} + (\mathbf{W}\mathbf{P})_i (\mathbf{a} \mathbf{b} | \mathbf{c} \mathbf{d})^{(m+1)} \\
 &+ \frac{a_i}{2\zeta} \left[(\mathbf{a} - 1_i \mathbf{b} | \mathbf{c} \mathbf{d})^{(m)} - \frac{\rho}{\zeta} (\mathbf{a} - 1_i \mathbf{b} | \mathbf{c} \mathbf{d})^{(m+1)} \right] \\
 &+ \frac{b_i}{2\zeta} \left[(\mathbf{a} \mathbf{b} - 1_i | \mathbf{c} \mathbf{d})^{(m)} - \frac{\rho}{\zeta} (\mathbf{a} \mathbf{b} - 1_i | \mathbf{c} \mathbf{d})^{(m+1)} \right] \\
 &+ \frac{c_i}{2(\zeta + \eta)} (\mathbf{a} \mathbf{b} | \mathbf{c} - 1_i \mathbf{d})^{(m+1)} \\
 &+ \frac{d_i}{2(\zeta + \eta)} (\mathbf{a} \mathbf{b} | \mathbf{c} \mathbf{d} - 1_i)^{(m+1)}
 \end{aligned} \tag{1}$$

is specified in a high-level form in C++ as shown in Listing 1.

- Second, the goal of the compiler is to make possible optimizing the code for existing and future computer hardware. For example, the ongoing trend in processors is to support single instruction multiple data (SIMD) parallelism. To effectively take advantage of the SIMD units The LIBINT compiler can generate vectorized code at user's request. As the hardware evolves, manual code reengineering can be avoided – only the compiler needs to be modified.

Listing 1: Example specification of an Obara-Saika recurrence relation in LIBINT compiler (see `src/bin/libint2/vrr.ll.twoprep.ll.h`). The corresponding mathematical expression is shown in Eq. (1)

```

auto ABCD_m = factory.make_child(a,b,c,d,m);
auto ABCD_mpl = factory.make_child(a,b,c,d,m+1);
expr_ = Vector("PA")[dir] * ABCD_m + Vector("WP")[dir] * ABCD_mpl;

auto aml = a - _1;
if (exists(aml)) {
    auto AmlBCD_m = factory.make_child(aml,b,c,d,m);
    auto AmlBCD_mpl = factory.make_child(aml,b,c,d,m+1);
    expr_ += Scalar(a[dir]) * Scalar("oo2z") * (AmlBCD_m - Scalar("roz") * AmlBCD_mpl);
}

auto bml = b - _1;
if (exists(bml)) {

```

```

    auto ABm1CD_m = factory.make_child(a,bm1,c,d,m);
    auto ABm1CD_mp1 = factory.make_child(a,bm1,c,d,m+1);
    expr_ += Scalar(b[dir]) * Scalar("oo2z") * (ABm1CD_m - Scalar("roz") * ABm1CD_mp1);
}

auto cm1 = c - _1;
if (exists(cm1)) {
    auto ABCm1D_mp1 = factory.make_child(a,b,cm1,d,m+1);
    expr_ += Scalar(c[dir]) * Scalar("oo2ze") * ABCm1D_mp1;
}

auto dm1 = d - _1;
if (exists(dm1)) {
    auto ABCDm1_mp1 = factory.make_child(a,b,c,dm1,m+1);
    expr_ += Scalar(d[dir]) * Scalar("oo2ze") * ABCDm1_mp1;
}

```

There are also drawbacks to the compiler-based LIBINT approach. First, the generated code can be fairly large and thus take a long time to compile. It is a relatively benign problem in practice. Also, the LIBINT compiler is a fairly complicated program; this limits the extent to which an average programmer can modify it.

LIBINT currently implements recursive schemes based on the Obara-Saika method,^{5,6} and Head-Gordon-Pople⁷ and Hamilton-Lindh variations thereof.^{8,9} Other recurrence relations can be easily implemented as needed.

Unlike version 1, which came as three separate interdependent libraries, version 2 of LIBINT comes as a single library configured at code-generation time. The following features of the library can be configured:

- support for four-center ERI, including optional support for two- and three-center ERIs,
- derivative level for ERI,
- support for some special two-body integrals for explicitly-correlated integrals, e.g. of $[\hat{T}_1, \exp(-\alpha r_{12}^2)]$,
- optimization features (whether shell-sets of integrals can be unrolled, whether to perform Common Subexpression Elimination, etc.),
- vectorization features,
- algorithmic features (evaluation strategy),
- API features (name prefix, FLOP counter, whether to accumulate target integrals, floating-point type, shell ordering of Cartesian basis functions),
- shared library support.

Depending on its configuration, the library may not lack some capabilities.

2 Overview of LIBINT's API

(**Note:** Depending on configuration, LIBINT may support computation of several types of integrals. This section describes the parts of the interface that deal with the evaluation of four-center two-body Coulomb integrals. Additional notes on computing other types of integrals are provided in later sections.)

LIBINT library is a low-level C++ code. C linking convention is adopted to enable interoperability with other languages, such as C and FORTRAN. For notes on how to use LIBINT library from FORTRAN code, see subsection 7.

LIBINT API consists of 3 major components: type definitions, function and variable prototypes, and C preprocessor macros. The API is described in the following header files:

- `libint2.h` – main header file, it includes most other headers (not generated).
- `libint2_intrinsic_types.h` – architecture-specific definitions for long integer types (not generated).
- `libint2_types.h` – definitions for the integral evaluator types (generated).

- `libint2_params.h` – C preprocessor macros for library features (generated).
- `libint2_iface.h` – prototypes for functions and data as well as some misc macros (generated).
- `boys.h` – engines for computation of the Boys function and related quantities (not generated).

`libint2.h` (and, optionally, `boys.h`) is the only header that must be included explicitly in the user code.

The preprocessor macros provided by `libint2.h` are necessary so that the user code can access the library configuration parameters. For example, the library can be configured at the code generation time to support any finite angular momentum (quantum numbers) of the basis functions. The user code must still test whether the maximum angular momentum present in the basis set exceeds the library limit.

The preprocessor value macros provided by `libint2.h` are listed in Listing 2. The most important macro, `LIBINT2_REALTYPE`, specifies the floating-point type used by the library for all computations (usually, `double`). The next macro, `LIBINT2_MAX_VECLLEN` specifies the maximum vector length supported by the library. For the scalar code this macro will be set to 1, whereas for the vectorized library this will have a value greater than 1. The next macro, `LIBINT2_API_PREFIX`, is used by the `LIBINT2_PREFIXED_NAME` function macro and should not be used explicitly (I list it here for completeness). The last three macros describe whether the library supports computation of the electron repulsion integrals, the maximum angular momentum of the basis functions supported for the ERI code (value of 3 corresponds to support of up to f functions, etc.), and the maximum order of the ERI derivatives which can be computed.

Listing 2: C preprocessor value macros provided by `libint2.h`. The angled brackets describe valid macro values.

```
#define LIBINT2_REALTYPE <C++ floating-point type>
#define LIBINT2_MAX_VECLLEN <positive integer>
#define LIBINT2_API_PREFIX <string>

/* ERI-specific macros */
#define LIBINT2_SUPPORT_ERI <0 or 1>
#define LIBINT2_MAX_AM_eri <nonnegative integer>
#define LIBINT2_DERIV_ERI_ORDER <nonnegative integer>
```

`LIBINT` API also specifies several functions and variables whose declarations are listed in Listing 3.

The first two functions perform static initialization and cleanup of the library. Thus `libint2_static_init` must be called before `LIBINT` is used and `libint2_static_cleanup` must be called after `LIBINT` is no longer needed (only one thread needs to call these functions).

The next two functions are used to initialize and cleanup the key data structure (*integral evaluator*) involved in the computation of the electron repulsion integrals. To initialize the integral evaluator, `libint2_init_eri` should be called with three arguments: 1) the pointer to the evaluator to be initialized (or to the first evaluator of an array of evaluators – see the notes on handling contracted Gaussians below); 2) the maximum angular momentum of basis functions this object will support (`max_am` will affect the memory requirements for the computation and therefore should be always set to the actual maximum value needed, not the maximum value supported by the library); 3) optional pointer to the scratch buffer which will be used to hold intermediate results. If the third argument is 0, then the call will dynamically allocate the needed space. If the user code needs to control memory allocation/deallocation, the scratch buffer needs to be allocated prior to the call. `libint2_need_memory_eri` can be used to compute the required size of the scratch buffer in units of `LIBINT2_REALTYPE`. Lastly, `libint2_build_eri` is a 4-dimensional array of pointers to functions that evaluate ERIs, e.g., `libint2_build_eri[1][0][2][0](&erieval)` will compute the $\langle ps|ds \rangle$ set using evaluator `erieval` (`erieval` must have been initialized with `libint2_init_eri`).

Listing 3: `LIBINT` API functions and data. $N = \text{LIBINT2_MAX_AM_ERI} + 1$.

```
void libint2_static_init();
void libint2_static_cleanup();

/* ERI-specific API */
void libint2_init_eri(Libint_eri_t* libint, int max_am, LIBINT2_REALTYPE* buf);
void libint2_cleanup_eri(Libint_eri_t* libint);
size_t libint2_need_memory_eri(int max_am);
void (*libint2_build_eri[N][N][N][N])(Libint_eri_t *);
```

LIBINT API described so far is very simple. However, as you may have noticed, there has been no mention of where the basis set data is stored. This is because LIBINT does not maintain the basis set information. As part of LIBINT's philosophy to provide the leanest possible code, the user code is in charge of precomputing basis set data and Boys function values and then feeding it to the evaluator object of type `Libint_eri_t`. (Note that since currently all integral evaluations in LIBINT use the same evaluator object type, they are all just typedefs to the common type `Libint_t`, henceforth we will use both interchangeably).

Listing 4: Definition of the LIBINT integral evaluator type.

```
typedef struct {
    _aB_s_0_s_1_TwoERep_s_0_s_1_Ab_up_0[VECLEN];
    _aB_s_0_s_1_TwoERep_s_0_s_1_Ab_up_1[VECLEN];
    _aB_s_0_s_1_TwoERep_s_0_s_1_Ab_up_2[VECLEN];
    _aB_s_0_s_1_TwoERep_s_0_s_1_Ab_up_3[VECLEN];
    _aB_s_0_s_1_TwoERep_s_0_s_1_Ab_up_4[VECLEN];
    /* and so on until 4 * LIBINT2_MAX_AM_ERI */

    LIBINT2_REALTYPE WP_x[VECLEN], WP_y[VECLEN], WP_z[VECLEN];
    LIBINT2_REALTYPE WQ_x[VECLEN], WQ_y[VECLEN], WQ_z[VECLEN];
    LIBINT2_REALTYPE PA_x[VECLEN], PA_y[VECLEN], PA_z[VECLEN];
    LIBINT2_REALTYPE QC_x[VECLEN], QC_y[VECLEN], QC_z[VECLEN];
    LIBINT2_REALTYPE AB_x[VECLEN], AB_y[VECLEN], AB_z[VECLEN];
    LIBINT2_REALTYPE CD_x[VECLEN], CD_y[VECLEN], CD_z[VECLEN];

    LIBINT2_REALTYPE oo2z[VECLEN];
    LIBINT2_REALTYPE oo2e[VECLEN];
    LIBINT2_REALTYPE oo2ze[VECLEN];
    LIBINT2_REALTYPE roz[VECLEN];
    LIBINT2_REALTYPE roe[VECLEN];

    LIBINT2_REALTYPE* stack; /* used internally */
    LIBINT2_REALTYPE* vstack; /* used internally */
    LIBINT2_REALTYPE* targets[LIBINT2_MAX_NTARGETS_eri];
    int veclen;

    LIBINT2_UINT_LEAST64* nflops;
    int zero_out_targets;
} Libint_eri_t;
```

The definition of `Libint_eri_t` is shown in Listing 4. `Libint_eri_t` is a C structure, i.e., its members can be manipulated directly. This is done to allow direct manipulation of `Libint_eri_t` from Fortran and other languages. Note that only ERI-specific parts of the actual definition `Libint_eri_t` are shown in Listing 4. Currently, the same data structure is used for all types of evaluators (i.e. for ERI evaluators and Gaussian geminal type evaluators), i.e. `Libint_eri_t` contains also members which are only used for evaluation of Gaussian geminal integrals also. In the future the evaluator types will be generated automatically and will be specific to each type of computation.

Let's look in detail at the members of `Libint_eri_t` which must be precomputed before calling the relevant `libint2_builderi` function:

- `_aB_s_0_s_1_TwoERep_s_0_s_1_Ab_up_m` – values of auxiliary primitive integrals $(00|00)^{(m)}$ (Eq. (25)) for $0 \leq m \leq \lambda(\mathbf{a}) + \lambda(\mathbf{b}) + \lambda(\mathbf{c}) + \lambda(\mathbf{d}) + C$, where $C = 0$ when computing ERIs, $C = 1$ when computing first derivative ERIs, etc.
- `AB_i`, `CD_i` – cartesian components of vectors $\mathbf{AB} \equiv \mathbf{A} - \mathbf{B}$ and $\mathbf{CD} \equiv \mathbf{C} - \mathbf{D}$.
- `PA_i`, `QC_i` – cartesian components of vectors $\mathbf{PA} \equiv \mathbf{P} - \mathbf{A}$ and $\mathbf{QC} \equiv \mathbf{Q} - \mathbf{C}$.
- `WP_i`, `WQ_i` – cartesian components of vectors $\mathbf{WP} \equiv \mathbf{W} - \mathbf{P}$ and $\mathbf{WQ} \equiv \mathbf{W} - \mathbf{Q}$.
- $oo2z - \frac{1}{2\zeta}$
- $oo2n - \frac{1}{2\eta}$

- $\text{oo2zn} - \frac{1}{2(\zeta + \eta)}$
- $\text{roz} - \frac{\rho}{\zeta}$
- $\text{ron} - \frac{\rho}{\eta}$

Most of these quantities are simple to evaluate. Evaluation of the Boys function needed to compute the auxiliary integrals $(\mathbf{00}|\mathbf{00})^{(m)}$ can be fairly complicated.¹⁰ LIBINT includes C++ classes that can evaluate Boys and related functions relatively efficiently (see `boys.h` header). An example of how to use LIBINT to compute Boys function is shown in Listing 5.

Listing 5: Computation of Boys function using LIBINT.

```
#include <libint2/boys.h>

// initialize Boys function engine to support m values up to mmax
// not thread-safe, should be constructed by main thread
libint2::FmEval_Chebyshev7 fmeval(mmax);

// double* Fm initialized somewhere else
// on return Fm[m], m=0..M, contains Fm(T)
// thread-safe provided each thread uses its own Fm
fmeval.eval(Fm, T, M);
```

Note that the evaluator object contains data that is depends on the Gaussian exponents. Hence for a shell set over contracted Gaussian functions data for each combination of primitive Gaussians will be kept in its own evaluator object, resulting in an array of evaluator objects necessary to evaluate contracted integrals. This is best illustrated by an example shown in Listing 6. This is perhaps the most significant change relative to the version 1 of the library, where the single evaluator object kept the data for all primitive combinations contributing to the given shell set.

Listing 6: Initialization and use of LIBINT with contracted basis functions.

```
libint2_static_init(); // static initialization (once per program)
std::vector<Libint_t> interval(contrdepth4); // array of contrdepth4 evaluators
// contrdepth4 = contrdepth^4, where contrdepth is
// the max contraction
// depth of Gaussians in the basis set

// initialize the array of evaluators .. this allocates the scratch array
// use libint2_need_memory_eri() to figure out how much memory is needed
libint2_init_eri(&interval[0], // ptr to the first evaluator in the array
               lmax,          // maximum angular momentum
               0);

// loop over (contracted) Gaussian shells
for(int s0=0; s0<nshell; ++s0) {
  for(int s1=0; s1<nshell; ++s1) {
    for(int s2=0; s2<nshell; ++s2) {
      for(int s3=0; s3<nshell; ++s3) {

        // decide whether to evaluate the integral (uniqueness, magnitude, etc.)
        // ..

        // loop over each primitive combination for this shell set
        int p0123 = 0;
        for(int p0=0; p0<nprim[s0]; ++p0) {
          for(int p1=0; p1<nprim[s1]; ++p1) {
            for(int p2=0; p2<nprim[s2]; ++p2) {
              for(int p3=0; p3<nprim[s3]; ++p3) {

                // optionally screen out primitive combinations

                // compute primitive data and put into interval[p0123]
                // ...
```

```

        ++p0123;
    }
}
}
}
// report the number of primitive combinations to libint
interval[0].contrdepth = p0123;

// evaluate the contracted integral shell set
libint2_build_eri(am[s0]][am[s1]][am[s2]][am[s3]](&interval[0]);

// grab the resulting shell set in interval[0].targets[0][...]
}
}
}

libint2_cleanup_eri(&interval[0]);
libint2_static_cleanup();

```

After the integrals have been built, they are placed somewhere in the scratch buffer. To recover their location, the array of pointers `targets` is provided, e.g., the computed ERI shell-set is located at `target[0]`. ERI evaluation produces only one shell-set of integrals, but other types of computations may produce several shell-sets of integrals at a time, e.g., usually all 12 derivative ERI integrals are computed at the same time. That's why `target` is an array of pointers, not a pointer.

Shell-sets of integrals contain integrals in “row major” order.¹¹ For example, if the number of functions in each shell is n_a , n_b , n_c , and n_d , respectively, then the integral $(ab|cd)$ is found at position $abcd = ((n_b + b)n_c + c)n_d + d$.

The rest of `Libint_eri_t` is used to control various aspects of its behavior:

- `veclength` is used in vectorized computation of integrals.
- `nflops` is used to count the total number of FLOPs (the library must have been configured with `--enable-flop-counter`).
- `zero_out_targets` is used to zero out the target integral buffers. This is only useful if `LIBINT` was configured with `--enable-accum-ints`.

Note that currently the `LIBINT` compiler minimizes the amount of code it generates by taking advantage of the permutational symmetry of the integrals. This means that only certain combinations of the angular momenta can be handled. In standard configuration `LIBINT` can evaluate a shell quartet $(\mathbf{ab}|\mathbf{cd})$ if $\lambda(\mathbf{a}) \geq \lambda(\mathbf{b})$, $\lambda(\mathbf{c}) \geq \lambda(\mathbf{d})$, and $\lambda(\mathbf{c}) + \lambda(\mathbf{d}) \geq \lambda(\mathbf{a}) + \lambda(\mathbf{b})$. (There is also the ordering used by `ORCA` program for which `LIBINT` can be configured — it will not be discussed here). If one needs to compute a quartet that doesn't conform to the rule, e.g. of type $(pf|sd)$, permutational symmetry of integrals can be utilized to compute such quartet:

$$(pq|rs) = (pq|sr) = (qp|rs) = (qp|sr) = (rs|pq) = (rs|qp) = (sr|pq) = (sr|qp) \quad (2)$$

In the case of $(pf|sd)$ shell quartet, one computes quartet $(ds|fp)$ instead, and then permutes function indices back to obtain the desired $(pf|sd)$.

The final integrals that `LIBINT` computes are not normalized. The best way to include the normalization is to scale the auxiliary integrals $(\mathbf{00}|\mathbf{00})^{(m)}$ by the normalization factors. However, Gaussians in a shell of angular momentum > 1 have different normalization factors. Usually the convention is to unit-normalize only the functions which have all quanta along one Cartesian direction, e.g., d_{xx} , f_{xxx} , etc. Some programs (e.g., `GAMESS`) require all functions to be unit-normalized; this can be achieved by scaling the final integrals.

3 Using `LIBINT` to compute non-Coulomb integrals

As elegantly shown by Ahlrichs,¹² Obara-Saika and related schemes can also be used to compute many two-body integrals with spherically-symmetric kernels $f(r_{12})$. Such integrals appear in, for example, explicitly correlated

R12/F12 methods or in range-separated form of Kohn-Sham density functional theory. To compute such integrals, instead of the Coulomb-kernel $(\mathbf{00}|\mathbf{00})^{(m)}$ integrals related to the Boys function by Eq. 28 LIBINT needs to be fed the corresponding kernel-specific $(\mathbf{00}|\mathbf{00})^{(m)}$ integrals related to the kernel-specific $G_m(\rho, T)$ function of Ahlrichs. Explicit expressions for G_m for several important kernels are given in Section 5 of Ref.¹². For example, for the Coulomb kernel $G_m(\rho, T)$ are related to the Boys function as (Eq. (39) of Ahlrichs paper):

$$G_0(\rho, T) = \frac{2\pi}{\rho} F_m(T). \quad (3)$$

Hence Eq. (28) changes to

$$(\mathbf{00}|\mathbf{00})^{(m)} = G_m(\rho, \rho|\mathbf{PQ}|^2) \sqrt{\frac{\rho^3}{\pi^3}} S_{12} S_{34} C_1 C_2 C_3 C_4. \quad (4)$$

where the overlaps S_{12} and S_{34} are defined in Eqs. (26) and (27), **not** the overlaps from Ahlrichs' paper.

LIBINT supports computation of the G_m function for the important case of a contracted Gaussian kernel that appears in explicitly correlated methods:

$$f_k(r_{12}) = r_{12}^k \sum_{i=1}^n c_i \exp(-\alpha_i r_{12}^2), \quad k = -1, 0, 2. \quad (5)$$

Example code is shown in Listing 7.

Listing 7: Computation of the G_m function for the contracted Gaussian kernel using LIBINT.

```
#include <libint2/boys.h>

// initialize Gaussian Gm function engine to support m values up to mmax
// k specifies the exponent of r_{12} in the kernel (0, -1, and 2 supported)
libint2::GaussianGmEval<double, k> gmeval(mmax, 1e-15); // 1e-15 is the desired precision

// vector of {exponent, coefficient} pairs = contracted Gaussian kernel
std::vector<std::pair<double, double> > gauss;

// double* Gm initialized somewhere else
// on return Gm[m], m=0..M, contains Gm(rho, T)
gmeval.eval(Gm, rho, T, M, gauss);
```

4 Using LIBINT to compute geometrical derivatives of integrals.

LIBINT can also evaluate geometrical derivatives of two-body integrals with respect to basis function positions. One shell set of four-center two-body integrals $(\mathbf{ab}|\mathbf{cd})$ has total of 12 first-order geometrical derivatives:

$$\frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial A_i}, \frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial B_i}, \frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial C_i}, \frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial D_i} : \quad i \in \{x, y, z\}$$

and $12 \times 12 = 144$ second-order derivatives, although only $\frac{12(12+1)}{2} = 78$ of those are unique because of permutation symmetry with respect to the order of taking the derivative:

$$\begin{aligned} & \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial A_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial B_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial C_i \partial C_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial D_i \partial D_j} : \quad i \leq j \in \{x, y, z\} \\ & \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial B_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial C_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial D_j}, \\ & \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial C_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial D_j}, \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial C_i \partial D_j} : \quad i, j \in \{x, y, z\} \end{aligned}$$

The translational invariance of the integral can be used to eliminate derivatives involving one of the centers (as a matter of convention, LIBINT skips the third center, **C**). This allows to eliminate 3 of the 12 first-order derivatives.

$$\frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial C_i} = -\frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial A_i} - \frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial B_i} - \frac{\partial(\mathbf{ab}|\mathbf{cd})}{\partial D_i} \quad i \in \{x, y, z\} \quad (6)$$

and 33 of the 78 second-order derivatives

$$\frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial C_j} = -\frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial A_j} - \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial B_j} - \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial D_j} \quad i, j \in \{x, y, z\} \quad (7)$$

$$\begin{aligned} \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial C_i \partial C_j} = & \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial A_j} + \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial C_j} + \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial D_j} \\ & \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_j \partial B_i} + \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_j \partial C_j} + \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_j \partial D_j} \\ & \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_j \partial D_i} + \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_j \partial D_i} + \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial D_j \partial D_j} \quad i \leq j \in \{x, y, z\} \end{aligned} \quad (8)$$

$$\frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial C_j} = -\frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_j \partial B_i} - \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial B_j} - \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial D_j} \quad i, j \in \{x, y, z\} \quad (9)$$

$$\frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial C_i \partial D_j} = -\frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial A_i \partial D_j} - \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial B_i \partial D_j} - \frac{\partial^2(\mathbf{ab}|\mathbf{cd})}{\partial D_i \partial D_j} \quad i, j \in \{x, y, z\} \quad (10)$$

$$(11)$$

Here's an easy way to understand how many derivatives are permutationally and translationally invariant: translational invariance allows to omit one center from consideration, hence we are left with 4-1=3 centers and $3 \times 3 = 9$ first-order geometrical derivatives; for the k th-order derivatives the order of differentiation does not matter, hence we are left with $(9+k)!/9!k!$ unique derivatives, i.e. $9 \times 10/2 = 45$ second-order derivatives.

To minimize the operation count, LIBINT evaluates a complete set of unique shell-set derivatives at once. On return, `Libint_t::target[i]` will contain the i -th set of derivative integrals. For the first-order derivatives sets $i = 0..2$ corresponds to the derivative with respect to $A_x..A_z$, respectively, $i = 3..5 = B_x..B_z$, and $i = 6..8 = D_x..D_z$; derivatives with respect to coordinates $C_x..C_z$ can be recovered using Eq. (6). Similarly, for the second-order derivatives `Libint_t::target[ij]` will contain ij -th set of derivative integrals, where ij is the composite "upper-triangle" index: $ij = i \times (19-i)/2 + (j-i)$, $0 \leq i \leq j < 9$. For example, `Libint_t::target[29]` thus contains the $\partial^2/\partial B_x \partial D_z$ derivative set ($ij = 29 \rightarrow i = 3, j = 8$).

The additional types of "compute" functions specific to the geometric derivatives are:

```
extern void (*libint2_build_eri1[5][5][5][5])(Libint_t *);
extern void (*libint2_build_eri2[4][4][4][4])(Libint_t *);
```

The former refers to functions which compute first derivative ERIs, and the second refers to functions which compute second derivative ERIs. The dimensions of each array are determined by the following 2 configure-time macros defined in `libint2.params.h`:

```
#define LIBINT2_MAX_AM_ERI1 5
#define LIBINT2_MAX_AM_ERI2 4
```

(the actual values all depend on how the library was configured).

Each derivative shell set is identical in structure to a nondifferentiated shell set, i.e. individual integrals are arranged in a row major order. Normalization convention for the derivative integrals is the same as for the regular ERIs.

5 Using LIBINT to compute three- and two-center integrals

No notes yet — see `tests/eri/test_eri.cc` for a working example of how to compute 2- and 3-center integrals and their geometrical derivatives.

6 Example: using libint to compute four-center two-body Coulomb integrals

A C++ function that uses LIBINT to evaluate four-center two-body Coulomb integrals over *primitive* (non-contracted) Gaussians is shown in Listing 8.

Listing 8: Using LIBINT from C++.

```
#include <libint2.h>
#include <libint2/boys.h>

#include <algorithm>
#include <iostream>

using namespace std;
using namespace libint2;

/** This function evaluates ERI over 4 primitive Gaussian shells.
    See tests/eri/test.cc for an example of how to deal with
    contracted Gaussians.

    For simplicity, many details are omitted here, e.g. normalization.
*/
void compute_eri(unsigned int am1, double alpha1, double A[3], unsigned int am2,
                 double alpha2, double B[3], unsigned int am3, double alpha3,
                 double C[3], unsigned int am4, double alpha4, double D[3]) {
    // I will assume that libint2_static_init() has been called elsewhere!

    //-----
    // ----- the code in this section would usually be placed outside this
    // function -----
    //          to occur once per calculation, not every time this function is
    //          called
    // - allocate ERI evaluator object
    Libint_eri_t erieval;
    const unsigned int max_am = max(max(am1, am2), max(am3, am4));
    libint2_init_eri(&erieval, max_am, 0);
#ifdef LIBINT_CONTRACTED_INTS
    // if have support for contracted integrals, set the contraction length to 1
    erieval.contrdepth = 1;
#endif
    // - initialize Boys function evaluator
    FmEval_Chebyshev7 fmeval(max_am);
    //-----

    //
    // Compute requisite data -- many of these quantities would be precomputed
    // for all nonnegligible shell pairs somewhere else
    //
    const double gammap = alpha1 + alpha2;
    const double Px = (alpha1 * A[0] + alpha2 * B[0]) / gammap;
    const double Py = (alpha1 * A[1] + alpha2 * B[1]) / gammap;
    const double Pz = (alpha1 * A[2] + alpha2 * B[2]) / gammap;
    const double PAx = Px - A[0];
    const double PAy = Py - A[1];
    const double PAz = Pz - A[2];
    const double PBx = Px - B[0];
    const double PBy = Py - B[1];
    const double PBz = Pz - B[2];
    const double AB2 = (A[0] - B[0]) * (A[0] - B[0]) +
                      (A[1] - B[1]) * (A[1] - B[1]) +
                      (A[2] - B[2]) * (A[2] - B[2]);

    erieval.PA_x[0] = PAx;
    erieval.PA_y[0] = PAy;
    erieval.PA_z[0] = PAz;
    erieval.AB_x[0] = A[0] - B[0];
```

```

erieval.AB_y[0] = A[1] - B[1];
erieval.AB_z[0] = A[2] - B[2];
erieval.oo2z[0] = 0.5 / gammap;

const double gammaq = alpha3 + alpha4;
const double gammapq = gammap * gammaq / (gammap + gammaq);
const double Qx = (alpha3 * C[0] + alpha4 * D[0]) / gammaq;
const double Qy = (alpha3 * C[1] + alpha4 * D[1]) / gammaq;
const double Qz = (alpha3 * C[2] + alpha4 * D[2]) / gammaq;
const double QCx = Qx - C[0];
const double QCy = Qy - C[1];
const double QCz = Qz - C[2];
const double QDx = Qx - D[0];
const double QDy = Qy - D[1];
const double QDz = Qz - D[2];
const double CD2 = (C[0] - D[0]) * (C[0] - D[0]) +
                  (C[1] - D[1]) * (C[1] - D[1]) +
                  (C[2] - D[2]) * (C[2] - D[2]);

erieval.QC_x[0] = QCx;
erieval.QC_y[0] = QCy;
erieval.QC_z[0] = QCz;
erieval.CD_x[0] = C[0] - D[0];
erieval.CD_y[0] = C[1] - D[1];
erieval.CD_z[0] = C[2] - D[2];
erieval.oo2e[0] = 0.5 / gammaq;

const double PQx = Px - Qx;
const double PQy = Py - Qy;
const double PQz = Pz - Qz;
const double PQ2 = PQx * PQx + PQy * PQy + PQz * PQz;
const double Wx = (gammap * Px + gammaq * Qx) / (gammap + gammaq);
const double Wy = (gammap * Py + gammaq * Qy) / (gammap + gammaq);
const double Wz = (gammap * Pz + gammaq * Qz) / (gammap + gammaq);

erieval.WP_x[0] = Wx - Px;
erieval.WP_y[0] = Wy - Py;
erieval.WP_z[0] = Wz - Pz;
erieval.WQ_x[0] = Wx - Qx;
erieval.WQ_y[0] = Wy - Qy;
erieval.WQ_z[0] = Wz - Qz;
erieval.oo2ze[0] = 0.5 / (gammap + gammaq);
erieval.roz[0] = gammapq / gammap;
erieval.roe[0] = gammapq / gammaq;

double K1 = exp(-alpha1 * alpha2 * AB2 / gammap);
double K2 = exp(-alpha3 * alpha4 * CD2 / gammaq);
double pfac =
    2 * pow(M_PI, 2.5) * K1 * K2 / (gammap * gammaq * sqrt(gammap + gammaq));

//
// evaluate Boys function F_m for all m in [0,am]
//
unsigned int am = am1 + am2 + am3 + am4;
double* F = new double[am + 1];
fmeval.eval(F, PQ2 * gammapq, am);

// (00|00)^m = pfac * F_m
erieval.LIBINT_T_SS_EREP_SS(0)[0] = pfac * F[0];
erieval.LIBINT_T_SS_EREP_SS(1)[0] = pfac * F[1];
erieval.LIBINT_T_SS_EREP_SS(2)[0] = pfac * F[2];
erieval.LIBINT_T_SS_EREP_SS(3)[0] = pfac * F[3];
erieval.LIBINT_T_SS_EREP_SS(4)[0] = pfac * F[4];
// etc.

// compute ERIs
libint2_build_eri[am1][am2][am3][am4](&erieval);

```

```

// Print out the integrals
const double* eri_shell_set = erieval.targets[0];
const unsigned int n1 = (am1 + 1) * (am1 + 2) / 2;
const unsigned int n2 = (am2 + 1) * (am2 + 2) / 2;
const unsigned int n3 = (am3 + 1) * (am3 + 2) / 2;
const unsigned int n4 = (am4 + 1) * (am4 + 2) / 2;
for (int a = 0; a < n1; a++) {
    for (int b = 0; b < n2; b++) {
        for (int c = 0; c < n3; c++) {
            for (int d = 0; d < n4; d++) {
                cout << "a = " << a << "b = " << b << "c = " << c << "d = " << d
                    << "(ab|cd) = " << *eri_shell_set;
                ++eri_shell_set;
            }
        }
    }
}

// ----- like the code at the beginning, this usually goes outside this
// function -----
libint2_cleanup_eri(&erieval);
}

```

To see how to use LIBINT for efficient computation of integrals over *contracted* basis functions refer to the sample code found in `tests/eri` directory.

7 Notes on using LIBINT from Fortran

7.1 Modern Fortran

Codes that compile with Fortran standard 2003 or later can easily access LIBINT via its Fortran bindings. The Fortran API is located in the `fortran` subdirectory of LIBINT and is organised as follows:

- `libint2_types.f.h` – definitions for the integral evaluator types (generated)
- `libint.f.F90` – prototypes for functions and data (not generated)
- `fortran_example.F90` – Fortran example: four-center two-body Coulomb integrals over primitive Gaussians and derivatives (not generated)

The Fortran API currently has support for two-, three- and four-center Coulomb integrals and its derivatives. Some parts of LIBINT C++ API that do not provide a C interface are missing - most notably the Boys function engine. For LIBINT2_REALTYPE only `float` and `double` (`c_float` and `c_double` in Fortran) are currently supported. If specific functions of Libint are missing (integrals other than Coulomb), it is easy to contribute them to LIBINT (`libint.f.F90`), or to write Fortran bindings for these in your own code.

The provided Fortran bindings are an exact translation of the C API. The definition of the Fortran derived type `Libint_t` is shown in Listing 9. The only notable difference to the corresponding C struct (Listing 4) is that variable names starting with an underscore in C are prefixed with a letter `f` in Fortran. Refer to Listing 10 for Fortran prototypes of procedures and data. Note that the index order of multidimensional arrays is reverse in fortan compared to C due to row-major vs. column-major order such that e.g. `libint2_build_eri[am1][am2][am3][am4]` in C corresponds to `libint2_build_eri(am4, am3, am2, am1)` in Fortran.

Listing 9: Fortran definition of the LIBINT integral evaluator type.

```

type, bind(c) :: Libint_t
    real(LIBINT2_REALTYPE), dimension(VECLEN) :: f_aB_s__0__ElecPot_s__0__Ab__up_0
    real(LIBINT2_REALTYPE), dimension(VECLEN) :: f_aB_s__0__ElecPot_s__0__Ab__up_1
    real(LIBINT2_REALTYPE), dimension(VECLEN) :: f_aB_s__0__ElecPot_s__0__Ab__up_2
    real(LIBINT2_REALTYPE), dimension(VECLEN) :: f_aB_s__0__ElecPot_s__0__Ab__up_3
    real(LIBINT2_REALTYPE), dimension(VECLEN) :: f_aB_s__0__ElecPot_s__0__Ab__up_4
    ! and so on until 4 * LIBINT2_MAX_AM_eri

```

```

real(LIBINT2_REALTYPE), dimension(VECLEN) :: WP_x, WP_y, WP_z
real(LIBINT2_REALTYPE), dimension(VECLEN) :: WQ_x, WQ_y, WQ_z
real(LIBINT2_REALTYPE), dimension(VECLEN) :: PA_x, PA_y, PA_z
real(LIBINT2_REALTYPE), dimension(VECLEN) :: QC_x, QC_y, QC_z
real(LIBINT2_REALTYPE), dimension(VECLEN) :: AB_x, AB_y, AB_z
real(LIBINT2_REALTYPE), dimension(VECLEN) :: CD_x, CD_y, CD_z

real(LIBINT2_REALTYPE), dimension(VECLEN) :: oo2z
real(LIBINT2_REALTYPE), dimension(VECLEN) :: oo2e
real(LIBINT2_REALTYPE), dimension(VECLEN) :: oo2ze
real(LIBINT2_REALTYPE), dimension(VECLEN) :: roz
real(LIBINT2_REALTYPE), dimension(VECLEN) :: roe

type(c_ptr) :: stack
type(c_ptr) :: vstack
type(c_ptr), dimension(LIBINT2_MAX_NTARGETS_eri) :: targets
integer(c_int) :: vecLen

type(c_ptr) :: nflops
integer(c_int) :: zero_out_targets
end type

```

Listing 10: LIBINT Fortran API procedures and data.

```

subroutine libint2_static_init() bind(C)
end subroutine

subroutine libint2_static_cleanup() bind(C)
end subroutine

! ERI-specific API

subroutine libint2_init_eri(libint, max_am, buf) bind(C)
  type(libint_t), dimension(*) :: libint
  integer(kind=c_int), value :: max_am
  type(c_ptr), value :: buf
end subroutine

subroutine libint2_cleanup_eri(libint) bind(C)
  type(libint_t), dimension(*) :: libint
end subroutine

function libint2_need_memory_eri(max_am) bind(C)
  integer(kind=c_int), value :: max_am
  integer(kind=c_size_t) :: libint2_need_memory_eri
end function

type(c_funptr), &
  dimension(0:LIBINT2_MAX_AM_eri, 0:LIBINT2_MAX_AM_eri, &
    0:LIBINT2_MAX_AM_eri, 0:LIBINT2_MAX_AM_eri), &
  bind(c) :: libint2_build_eri

```

Using LIBINT from Fortran requires the Fortran builtin `iso_c_binding` module in order to ensure compatibility between Fortran and C data types. Fortran pointers are not directly interoperable with C pointers, therefore special datatypes for C pointers (`c_ptr` and `c_funptr`) and methods from `iso_c_binding` to convert to Fortran pointers (`c_f_pointer` and `c_f_procpointer`) are needed. If your code should be compatible with different configurations of libint compiler, you need to use conditional compilation based on the macros provided in `libint2/config.h` and `libint2/libint2.params.h` such as the macros `LIBINT2_MAX_AM_eri` and `LIBINT2_CONTRACTED_INTS` (for their use see listings 9 and 11).

An example subroutine that demonstrates how to actually calculate integrals using LIBINT is provided in Listing 11. This example is a translation of Listing 8 to Fortran and evaluates four-center two-body Coulomb integrals over primitive Gaussians. See `fortran_example.F90` for an example of how to deal with contracted

Gaussians and derivatives.

Listing 11: Using LIBINT from Fortran.

```
! include libint configuration-specific macros (may not be needed)
#include <libint2/config.h>
#include <libint2/libint2_params.h>

module libint_f_example

  USE iso_c_binding ! C interoperability
  USE libint_f      ! libint fortran bindings

  implicit none

contains

  ! This subroutine evaluates ERI over 4 primitive Gaussian shells.
  ! See compute_eri_f in fortran/libint_f.F90 for an example of how
  ! to deal with contracted Gaussians.
  subroutine compute_eri(am1, alpha1, A, &
                        am2, alpha2, B, &
                        am3, alpha3, C, &
                        am4, alpha4, D)

    integer, parameter :: dp = c_double ! C interoperable double type
    integer, parameter :: i = c_int     ! C interoperable int type

    integer(kind=i), intent(in) :: am1, am2, am3, am4
    real(kind=dp), intent(in) :: alpha1, alpha2, alpha3, alpha4
    real(kind=dp), intent(in), dimension(3) :: A, B, C, D

    real(kind=dp) :: gammap, AB2, CD2, PQ2, gammaq, gammapq, K1, K2, pfac
    real(kind=dp), allocatable, dimension(:) :: F
    real(kind=dp), dimension(:), pointer :: eri_shell_set
    real(kind=dp), dimension(3) :: P, Q, QC, QD, PQ, PA, PB, W
    real(kind=dp), parameter :: pi = 4*atan(1.0_dp)
    integer(kind=i) :: n1, n2, n3, n4, na, nb, nc, nd, ishell, max_am, am

    type(libint_t), dimension(1) :: erieval

    procedure(libint2_build), pointer :: build_eri

    ! I will assume that libint2_static_init() has been called elsewhere!

    !
    ! initialize (usually outside this function)
    !

    max_am = maxval([am1, am2, am3, am4])
    call libint2_init_eri(erieval, max_am, C_NULL_PTR)

#if LIBINT_CONTRACTED_INTS
    ! if have support for contracted integrals, set the contraction length to 1
    erieval(1)%contrdepth = 1
#endif

    !
    ! Compute requisite data
    !

    gammap = alpha1 + alpha2
    P = (alpha1*A + alpha2*B)/gammap
    PA = P - A
    PB = P - B
    AB2 = sum((A - B)*(A - B))

    erieval(1)%PA_x(1) = PA(1)
```

```

erieval(1)%PA_y(1) = PA(2)
erieval(1)%PA_z(1) = PA(3)
erieval(1)%AB_x(1) = A(1) - B(1)
erieval(1)%AB_y(1) = A(2) - B(2)
erieval(1)%AB_z(1) = A(3) - B(3)
erieval(1)%oo2z(1) = 0.5_dp/gammap

gammaq = alpha3 + alpha4
gammapq = gammap*gammaq/(gammap + gammaq)
Q = (alpha3*C + alpha4*D)/gammaq
QC = Q - C
QD = Q - D
CD2 = sum((C - D)*(C - D))

erieval(1)%QC_x(1) = QC(1)
erieval(1)%QC_y(1) = QC(2)
erieval(1)%QC_z(1) = QC(3)
erieval(1)%CD_x(1) = C(1) - D(1)
erieval(1)%CD_y(1) = C(2) - D(2)
erieval(1)%CD_z(1) = C(3) - D(3)
erieval(1)%oo2e(1) = 0.5_dp/gammaq

PQ = P - Q
PQ2 = sum(PQ*PQ)
W = (gammap*P + gammaq*Q)/(gammap + gammaq)

erieval(1)%WP_x(1) = W(1) - P(1)
erieval(1)%WP_y(1) = W(2) - P(2)
erieval(1)%WP_z(1) = W(3) - P(3)
erieval(1)%WQ_x(1) = W(1) - Q(1)
erieval(1)%WQ_y(1) = W(2) - Q(2)
erieval(1)%WQ_z(1) = W(3) - Q(3)
erieval(1)%oo2ze(1) = 0.5_dp/(gammap + gammaq)

erieval(1)%roz(1) = gammapq/gammap
erieval(1)%roe(1) = gammaq/gammaq

K1 = exp(-alpha1*alpha2*AB2/gammap)
K2 = exp(-alpha3*alpha4*CD2/gammaq)
pfac = 2*pi**2.5_dp*K1*K2/(gammap*gammaq*sqrt(gammap + gammaq))

am = am1 + am2 + am3 + am4

!
! evaluate Boys function F_m for all m in [0, am]
!

! here we omit the evaluation of Boys function values F(m+1)
! these would be calculated externally since libint Boys function engine is not
! accessible from Fortran
allocate F(am + 1)
F(1) = ...
F(2) = ...
...
F(am + 1) = ...

if (am >= 0) &
    erieval(1)%f_aB_s__0__s__1__TwoPRep_s__0__s__1__Ab__up_0(1) = pfac*F(1)
if (am >= 1) &
    erieval(1)%f_aB_s__0__s__1__TwoPRep_s__0__s__1__Ab__up_1(1) = pfac*F(2)
if (am >= 2) &
    erieval(1)%f_aB_s__0__s__1__TwoPRep_s__0__s__1__Ab__up_2(1) = pfac*F(3)
if (am >= 3) &
    erieval(1)%f_aB_s__0__s__1__TwoPRep_s__0__s__1__Ab__up_3(1) = pfac*F(4)
if (am >= 4) &
    erieval(1)%f_aB_s__0__s__1__TwoPRep_s__0__s__1__Ab__up_4(1) = pfac*F(5)
! etc.

```

```

!
! compute ERIs
!

! convert C function pointer libint2_build_eri(am4, am3, am2, am1) to
! Fortran procedure pointer build_eri
call c_f_procpointer(libint2_build_eri(am4, am3, am2, am1), build_eri)

! call function to compute ERIs
call build_eri(erieval)

!
! Print out the integrals
!

n1 = (am1 + 1)*(am1 + 2)/2
n2 = (am2 + 1)*(am2 + 2)/2
n3 = (am3 + 1)*(am3 + 2)/2
n4 = (am4 + 1)*(am4 + 2)/2

! get C pointer to targets and convert to Fortran pointer eri_shell_set
call c_f_pointer(erieval(1)%targets(1), eri_shell_set, shape=[n1*n2*n3*n4])

ishell = 0
do na = 1, n1
  do nb = 1, n2
    do nc = 1, n3
      do nd = 1, n4
        ishell = ishell + 1
        print *, "a = ", na, &
          "b = ", nb, &
          "c = ", nc, &
          "d = ", nd, &
          "(ab|cd) = ", eri_shell_set(ishell)
      enddo
    enddo
  enddo
enddo

!
! cleanup (usually outside this function)
!

call libint2_cleanup_eri(erieval)
end subroutine

end module

```

7.2 Fortran 77

Codes written in Fortran 77 can not use the Fortran bindings included in `LIBINT`. In general, C functions can be easily called from Fortran 77 programs, but sharing data structures is not straightforward. Thus the main culprit is how to modify `Libint_eri_t` objects from Fortran. Older Fortran standards can access C data structures indirectly, via common blocks. An (non-working) example of how a Fortran subroutine can manipulate `Libint_eri_t` is shown in Listing 12. The common block `erieval` referenced in that Listing is created by declaring in C++ a global variable as follows:

```
extern Libint_eri_t erieval;
```

Listing 12: Accessing `Libint_eri_t` structure from a Fortran code.

```

c assuming that LIBINT2_REALTYPE is 8-bytes long
real(8) F0(1)
real(8) F1(1)

```



```

real(8) F2(1)
real(8) F3(1)
real(8) F4(1)
etc.
real(8) WP_x(1), WP_y(1), WP_z(1)
real(8) WQ_x(1), WQ_y(1), WQ_z(1)
real(8) PA_x(1), PA_y(1), PA_z(1)
real(8) QC_x(1), QC_y(1), QC_z(1)
real(8) AB_x(1), AB_y(1), AB_z(1)
real(8) CD_x(1), CD_y(1), CD_z(1)
real(8) oo2z(1), oo2e(1), oo2ze(1), roz(1), roe(1)
c in 64-bit environment pointers are 8-bytes long
c assuming LIBINT2_MAX_NTARGETS is 10
integer(8) targets(10)
integer(4) veclength
integer(8) nflops
integer(4) zero_out_targets
c common erieval represents the C object erieval
common/erieval/ F0, F1, ... , WP_x, WP_y, etc.

c now can access elements of erieval
AB_x[0] = Ax[0] - Bx[0]
etc.

```

The caveat of accessing C data structures from Fortran programs is that the members of the data structure must be declared in the common block in the exact order in which they appear in the definition of `Libint_eri_t`. The actual definition of `Libint_eri_t` in `libint2_types.h` must always be consulted.

Calling `LIBINT` functions from Fortran should be straightforward. Using the function pointer array `libint2_build_eri` may not be feasible in older Fortran standards. Actual function names must be used instead, i.e., a C++ expression

```
libint2_build_eri[1][0][2][0](&erival);
```

will be replaced with a Fortran expression (I'm not sure how to pass pointer to `erieval` to the function from Fortran!)

```
_aB_p__0__d__1___TwoERep_s__0__s__1___Ab__up_0()
```

where `_aB_p__0__d__1___TwoERep_s__0__s__1___Ab__up_0` is the name of the function to which `libint2_build_eri[1][0][2][0]` points.

Appendices

A Notation

Following Obara and Saika,⁵ we write an *unnormalized primitive Cartesian* Gaussian function centered at **A** as

$$\begin{aligned} \phi(\mathbf{r}; \zeta, \mathbf{n}, \mathbf{A}) &= (x - A_x)^{n_x} (y - A_y)^{n_y} (z - A_z)^{n_z} \\ &\times \exp[-\zeta(\mathbf{r} - \mathbf{A})^2], \end{aligned} \quad (12)$$

where **r** is the coordinate vector of the electron, ζ is the orbital exponent, and **n** is a set of non-negative integers. The sum of n_x , n_y , and n_z will be denoted $\lambda(\mathbf{n})$ and be referred to as the angular momentum or orbital quantum number of the Gaussian function. Hereafter **n** will be termed the angular momentum index. Henceforth, n_i will refer to the i -th component of **n**, where $i \in \{x, y, z\}$. Basic vector addition rules will apply to these vector-like triads of numbers, e.g. $\mathbf{n} + \mathbf{1}_x \equiv \{n_x + 1, n_y, n_z\}$.

A set of $(\lambda(\mathbf{n}) + 1)(\lambda(\mathbf{n}) + 2)/2$ functions with the same $\lambda(\mathbf{n})$, ζ , and centered at the common center but with different **n** form a *Cartesian shell*, or just a *shell*. For example, an *s* shell ($\lambda = 0$) has one function, a *p* shell ($\lambda = 1$) – 3 functions, etc. There is no unique choice for the order of functions in shells. The standard `LIBINT`

ordering is:

```

p : px, py, pz
d : dxx, dxy, dxz, dyy, dyz, dzz
f : fxxx, foxy, foxz, fxyy, fxyz, fxzz, fyyy, fyyz, fyzz, fzzz
etc.

```

In general, the following loop structure can be used to generate angular momentum indices in the canonical LIBINT order for all members of a shell of angular momentum `am`:

```

for(int i=0; i<=am; i++) {
  int nx = am - i; /* exponent of x */
  for(int j=0; j<=i; j++) {
    int ny = i-j; /* exponent of y */
    int nz = j; /* exponent of z */
  }
}

```

Other shell orderings are supported as well, e.g., those employed by the `intv3` engine in the `MPQC` program, the ordering used in the `GAMESS` program, or that in the `ORCA` program. These can be specified when the library is generated (see the `--with-cartgauss-ordering` configure flag). Support of a new ordering is trivial to implement. If your program relies on an ordering different from the above, please contact the author of `LIBINT`.

The normalization constant for a primitive Gaussian $\phi(\mathbf{r}; \zeta, \mathbf{n}, \mathbf{A})$

$$N(\zeta, \mathbf{n}) = \left[\left(\frac{2}{\pi} \right)^{3/4} \frac{2^{(\lambda(\mathbf{n}))} \zeta^{(2\lambda(\mathbf{n})+3)/4}}{[(2n_x - 1)!!(2n_y - 1)!!(2n_z - 1)!!]^{1/2}} \right] \quad (13)$$

A contracted Gaussian function is just a linear combination of primitive Gaussians (also termed *primitives*) centered at the same center \mathbf{A} and with the same momentum indices \mathbf{n} but with different exponents ζ_i :

$$\begin{aligned} \phi(\mathbf{r}; \zeta, \mathbf{C}, \mathbf{n}, \mathbf{A}) &= (x - A_x)^{n_x} (y - A_y)^{n_y} (z - A_z)^{n_z} \\ &\times \sum_{i=1}^M C_i \exp[-\zeta_i (\mathbf{r} - \mathbf{A})^2], \end{aligned} \quad (14)$$

Contracted Gaussians form shells the same way as primitives. The contraction coefficients \mathbf{C} already include normalization constants so that the resulting combination is properly normalized. Published contraction coefficients \mathbf{c} are linear coefficients for normalized primitives, hence the normalization-including contraction coefficients \mathbf{C} have to be computed from them as

$$C_i = c_i N(\zeta_i, \mathbf{n}) \quad (15)$$

and scaled further so that the self-overlap of the contracted function is 1:

$$\frac{\pi^{3/2} (2n_x - 1)!! (2n_y - 1)!! (2n_z - 1)!!}{2^{\lambda(\mathbf{n})}} \sum_{i=1}^M \sum_{j=1}^M \frac{C_i C_j}{(\zeta_i + \zeta_j)^{\lambda(\mathbf{n})+3/2}} = 1 \quad (16)$$

If sets of orbital exponents are used to form contracted Gaussians of one angular momentum only then this is called a *segmented* contraction scheme. If there is a set of exponents that forms contracted Gaussians of several angular momenta then such scheme is called *general* contraction. Examples of basis sets that include general contractions include Atomic Natural Orbitals (ANO) sets. `LIBINT` was not designed to handle general contractions very well. You should use either split general contractions into segments for each angular momentum (it's done for correlation consistent basis sets) or use basis sets with segmented contractions only.

An integral of a two-body operator $\hat{O}(\mathbf{r}_1, \mathbf{r}_2)$ over unnormalized primitive Cartesian Gaussians is written as

$$\int \phi(\mathbf{r}_1; \zeta_a, \mathbf{a}, \mathbf{A}) \phi(\mathbf{r}_2; \zeta_c, \mathbf{c}, \mathbf{C}) \hat{O}(\mathbf{r}_1, \mathbf{r}_2) \phi(\mathbf{r}_1; \zeta_b, \mathbf{b}, \mathbf{B}) \phi(\mathbf{r}_2; \zeta_d, \mathbf{d}, \mathbf{D}) d\mathbf{r}_1 d\mathbf{r}_2 \equiv (\mathbf{ab}|\hat{O}|\mathbf{cd}) \equiv \langle \mathbf{ac}|\hat{O}|\mathbf{bd} \rangle \quad (17)$$

A set of integrals $\{(\mathbf{ab}|\hat{O}(\mathbf{r}_1, \mathbf{r}_2)|\mathbf{cd})\}$ over all possible combinations of functions $\mathbf{a} \in \text{ShellA}$, $\mathbf{b} \in \text{ShellB}$, etc. will be termed a *shell-set*, or simply a *set*, of integrals. For example, a $(ps|sd)$ set consists of $3 \times 1 \times 1 \times 6 = 18$ integrals.

The following definitions have been used throughout this work:

$$\zeta = \zeta_a + \zeta_b \quad (18)$$

$$\eta = \zeta_c + \zeta_d \quad (19)$$

$$\rho = \frac{\zeta\eta}{\zeta + \eta} \quad (20)$$

$$\mathbf{P} = \frac{\zeta_a \mathbf{A} + \zeta_b \mathbf{B}}{\zeta} \quad (21)$$

$$\mathbf{Q} = \frac{\zeta_c \mathbf{C} + \zeta_d \mathbf{D}}{\eta} \quad (22)$$

$$\mathbf{W} = \frac{\zeta \mathbf{P} + \eta \mathbf{Q}}{\zeta + \eta} \quad (23)$$

The Boys function is defined as

$$F_m(T) = \int_0^1 dt \, t^{2m} \exp(-Tt^2) \quad (24)$$

Evaluation of integrals over functions of non-zero angular momentum starts with the *auxiliary* integrals over primitive *s*-functions defined as

$$(00|00)^{(m)} = 2F_m(\rho|\mathbf{PQ}|^2) \sqrt{\frac{\rho}{\pi}} S_{12} S_{34} \quad (25)$$

where $\mathbf{PQ} = \mathbf{P} - \mathbf{Q}$ and primitive overlaps S_{12} and S_{34} are computed as

$$S_{12} = \left(\frac{\pi}{\zeta}\right)^{3/2} \exp\left(-\frac{\zeta_a \zeta_b}{\zeta} |\mathbf{AB}|^2\right) \quad (26)$$

$$S_{34} = \left(\frac{\pi}{\eta}\right)^{3/2} \exp\left(-\frac{\zeta_c \zeta_d}{\eta} |\mathbf{CD}|^2\right) \quad (27)$$

In the evaluation of integrals over contracted functions it is convenient to use auxiliary integrals over primitives which include contraction and normalization factors of the target quartet $(\mathbf{ab}|\mathbf{cd})$:

$$(00|00)^{(m)} = 2F_m(\rho|\mathbf{PQ}|^2) \sqrt{\frac{\rho}{\pi}} S_{12} S_{34} C_1 C_2 C_3 C_4 \quad (28)$$

where the coefficients C_a , C_b , C_c , and C_d are normalization-including contraction coefficients (Eqs. (15) and (16)) for the first basis function out of each respective shell in the target shell of integrals.

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